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Lattice supersymmetry and topological field theory

Simon Catterall
Department of Physics, Syracuse University
Syracuse, NY 13244, USA
E-mail: smc@physics.syr.edu

Abstract: It is known that certain theories with extended supersymmetry can be discretized in such a way as to preserve an exact fermionic symmetry. In the simplest model of this kind, we show that this residual supersymmetric invariance is actually a BRST symmetry associated with gauge fixing an underlying local shift symmetry. Furthermore, the original lattice action is then seen to be entirely a gauge fixing term. The corresponding continuum theory has a topological field theory interpretation. We look, in detail, at one example - supersymmetric quantum mechanics which possesses two such BRST symmetries. In this case, we show that the lattice theory can be obtained by blocking out of the continuum in a carefully chosen background metric. Such a procedure will not change the Ward identities corresponding to the BRST symmetries since they correspond to topological observables. Thus, at the quantum level, the continuum BRST symmetry is preserved in the lattice theory. Similar conclusions are reached for the two-dimensional complex Wess-Zumino model and imply that all the supersymmetric Ward identities are satisfied *exactly* on the lattice. Numerical results supporting these conclusions are presented.

Keywords: Topological Field Theories, Lattice Quantum Field Theory, Extended Supersymmetry.

Contents

1. Introduction

Supersymmetric field theories are interesting both from phenomenological and theoretical points of view - their improved U.V behavior offers the hope of resolving the gauge hierarchy problem and they arise naturally as low energy limits of string and M-theory. Most of the interesting physics of such theories lies in non-perturbative regimes. Discretization on a space-time lattice appears to provide a natural way to study such theories and considerable effort has gone into formulating such lattice supersymmetric theories [1]. Unfortunately supersymmetry is typically broken at the classical level in such models. At the quantum level the absence of such a symmetry leads to an effective action containing relevant SUSY-violating interactions. To achieve a supersymmetric continuum limit then necessitates fine tuning the couplings to each of these operators. In most situations this is prohibitively difficult.

In an previous paper [2] we advocated a different approach — try to preserve a subset of the full supersymmetry in the lattice model. A similar approach has also been adopted in [3]. Numerical simulations of two models where this idea can be implemented explicitly lend strong support to the idea that preserving some subset of the continuum supersymmetry transformations can indeed protect the lattice model from the dangerous radiative corrections that generically plague discretizations of supersymmetric field theories [2, 4]. In this paper we offer a proof of this statement for the case of supersymmetric quantum mechanics and the two-dimensional complex Wess-Zumino model. Indeed, our result is much stronger - by careful choice of the lattice action we can show that there are *no* cut-off effects in the lattice supersymmetric Ward identities even for supersymmetries which are broken at the classical level.

We will argue that the reason that these lattice models are well behaved is that they are related to Witten-type continuum topological field theories [5]. Such theories are constructed using a nilpotent symmetry formed from elements of the original supersymmetry. A key feature of such theories is that they contain observables whose expectation values are independent of the metric. After the partition function itself, the simplest examples of such observables are the Ward identities corresponding to the nilpotent symmetry. This metric independence plays a crucial role in allowing us to establish a direct link between the continuum and lattice theories and allows the latter to possess rather remarkable properties.

In the next section we introduce a simple lattice model which exhibits an exact fermionic symmetry and show that this symmetry can be viewed as a BRST symmetry following from fixing a local topological symmetry. We then show how this topological model can be used to construct a lattice regularized theory of supersymmetric quantum mechanics [4]. In this latter case we show that the model is naturally associated with two independent BRST symmetries. In the continuum these two symmetries exhaust the original supersymmetries and show that supersymmetric quantum mechanics, when suitably restricted, has a topological field theory interpretation.

For convenience we include a brief summary of the main features of such topological field theories. We then show how this continuum topological structure allows us to derive the lattice theory by integrating out the continuum fields in a carefully chosen background geometry. More precisely, we consider metrics which are continuous functions of a parameter β such that in the limit $\beta \to \infty$ a "lattice" structure is induced in the model. For any finite value of β we can perform an associated β-dependent change of variables in the continuum model which preserves the topological symmetries. Hence such a procedure ensures that any topological observable is *independent* of β. We show that indeed, a local, lattice theory is approached in the limit of large β. Furthermore, our construction then guarantees that the resulting lattice theory retains an element of supersymmetry at the quantum level.

The same analysis applied to the complex Wess Zumino model in two dimensions allows us to write down lattice actions with respect to which all supersymmetric Ward identities are satisfied exactly. Numerical results confirming these conclusions are presented. Finally we discuss the prospects for extending these ideas to more realistic models.

2. Simple example

Consider the simple model discussed in [2]

$$S = \frac{1}{2\alpha} N_i^2(x) + \overline{\psi}_i \frac{\partial N_i}{\partial x_j} \psi_j \tag{2.1}$$

which admits a fermionic symmetry

$$\begin{aligned} \delta x_i &= \psi_i \xi \\ \delta \overline{\psi}_i &= N_i \xi \\ \delta \psi_i &= 0 \,. \end{aligned}$$

Here, $N_i(x)$ is an *arbitrary* function of the scalar field x_i and ψ_i, $\overline{\psi}_i$ are real, independent grassmann variables. Notice that this transformation is nilpotent on-shell. Indeed, if we introduce an auxiliary (commuting) field B_i we can define a new action

$$S' = -\frac{1}{2}\alpha B_i^2 + N_i B_i + \overline{\psi}_i \frac{\partial N_i}{\partial x_j}\psi_j \,. \tag{2.2}$$

If we integrate over B_i (along the imaginary axis) we just recover the original action S. This new action also has an invariance

$$\begin{aligned} \delta_{(1)} x_i &= \psi_i \xi \\ \delta_{(1)} \overline{\psi}_i &= B_i \xi \\ \delta_{(1)} \psi_i &= 0 \\ \delta_{(1)} B_i &= 0 \,. \end{aligned}$$

The significance of the subscript labeling the action and symmetry variation will be become apparent later. It is easy to see that this new transformation is nilpotent off-shell $\delta^2_{(1)}\Phi = 0$ for any of the fields $\Phi = \{\psi, \overline{\psi}, x, B\}$. More importantly it is clear that the new action S' is nothing but the variation of another function

$$S'\xi = \delta_{(1)}\left(\overline{\psi}_i\left(N_i - \frac{1}{2}\alpha B_i\right)\right).$$

Thus we recognize our original invariance as a BRST invariance and our original action as nothing but a gauge fixing term! The topological origins of the lattice theory are made more manifest when it is realized that the local gauge symmetry which is being fixed to generate the BRST invariance is nothing but an arbitrary shift in the scalar field x_i. Imagine a classical theory depending on a scalar field x_i with trivial classical action $S(x) = 0$. Clearly this theory is invariant under a huge local symmetry - namely arbitrary shifts in the scalar field

$$x_i \to x_i + \epsilon_i \,.$$

To quantize we need to pick a gauge. One simple way to do is is to impose $N_i(x) = 0$ where $N_i(x)$ is some arbitrary function of the field x_i. Then the partition function will be

$$Z = \int dx_i \, \delta(N_i) \det\left(\frac{\partial N_i}{\partial x_j}\right).$$

If we represent the determinant using anticommuting ghosts and introduce a multiplier field B_i for the delta function we recover our simple model eq. (2.1) in $\alpha = 0$ gauge. The usual theorem associated with quantization of gauge theories allows us to relax this Landau-like gauge to a Feynman-like gauge with α non-zero without changing the expectation values of gauge invariant quantities.

Notice that the physical 'fermions' of the SUSY theory are to be identified with the ghosts of the gauge fixed scalar field theory. This observation is also at the heart of a fundamental difference between the topological field theory and its parent supersymmetric

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theory. The physical states of the former are required to be annihilated by the BRST charge – this ensures the absence of ghosts and negative norm states in the spectrum. The only states in the parent supersymmetric theory with this property are the vacuum states — those annihilated by the supercharges. Thus the topological theory is really only a small subsector of the supersymmetric theory concerned with the space of vacuum solutions of the latter. In this paper we build our lattice supersymmetric models in a way which makes explicit their topological field theory interpretation. This allows us to build lattice models which retain an element of supersymmetry. However our models will *not* be truly topological since we will drop the requirement that physical states are invariant under the BRST symmetry. Thus our models should be interpreted as genuine discretizations of continuum supersymmetric field theories. This relaxation of the physical conditions should be ok for low-dimensional models where the spin-statistics theorem is not very restrictive and there is no real distinction between ghost and physical fermion fields.

There is another simple way to see the the model written down in eq. (2.1) exhibits an unusual symmetry. If we imagine performing a change of variables in the path integral defining this theory according to $\eta_i = N_i(x)$ the jacobian of this transformation cancels the fermionic determinant and the partition function factorizes into a product of gaussians [6]

$$Z = \prod_i d\eta_i e^{-\frac{\eta^2}{2\alpha}} \,.$$

This resultant partition function is trivially invariant under the same local shift symmetry discussed earlier. A transformation of this type which cancels off the fermion determinant is called a Nicolai map and it should now be clear that the existence of a *local* Nicolai map can be attributed to the presence of a topological symmetry [5].

We now turn to the simplest application of these ideas - supersymmetric quantum mechanics

3. Supersymmetric quantum mechanics

Imagine now choosing the function $N_i(x) = N_i^{(1)}(x)$ corresponding to an action $S' = S_{(1)}$ where

$$N_i^{(1)}(x) = D^S_{ij} x_j + P'_i(x) \,.$$

In this expression D^S represents the symmetric difference operator and $P'_i(x)$ is some arbitrary polynomial in the lattice field x_i. To be concrete we can imagine a model with a single interaction coupling g of the form

$$P' = m x_i + m^W_{ij} x_j + g x_i^3 \,. \tag{3.1}$$

Notice that we are also free to add a Wilson mass term m^W to the potential to eliminate lattice doubles associated with the choice of lattice derivative operator D^S.

In this case we recognize our simple model as a lattice regularized version of supersymmetric quantum mechanics - a model well known to possess a topological field theory interpretation [5]. We discuss some of the generic features of such theories in the next section but suffice it to say here that such theories possess observables whose expectation values are metric independent. Clearly, on the lattice, there is no notion of continuum metric and so in a strict sense the lattice model cannot be said to be topological. However, the fact that the action is a BRST variation of a local function of the lattice fields clearly imposes strong restrictions on the quantum theory. Indeed we shall see that in this model certain symmetries which are broken at the level of the classical lattice action are restored in the full quantum theory.

Notice also that the partition function of this model (in $\alpha = 0$ gauge) just reduces to an integral over the set of field configurations satisfying the gauge condition. In the case of supersymmetric quantum mechanics this is just the moduli space of classical solutions to the equation of motion. On a circle these classical solutions are just a finite set of points $x_i = x_c$ where $P'(x_c) = 0$. Thus the partition function just reduces to a sum over the critical points of the potential $P(x)$. Notice that this solution is *independent* of the lattice cut-off — indeed it is the same result one would have gotten for the analogous continuum model.

Furthermore we can consider another gauge condition which corresponds to the same classical moduli space $N_i(x) = N_i^{(2)}(x)$ with

$$N_i^{(2)}(x) = D^S_{ij} x_j - P'(x)\,.$$

We can construct the action $S_{(2)}$ following from this gauge condition by the same procedure and furthermore after exchanging the roles of ghost ψ and antighost $\overline{\psi}$ we can easily see it only differs from $S_{(1)}$ by the addition of a simple operator $C(g)$

$$S_{(2)} = S_{(1)} - 2C\,.$$

The operator $C = D^S_{ij} x_j P'_i(x)$ and corresponds to the integral of a total derivative term in the continuum. On the lattice it will be non-zero if $P'(x)$ contains nonlinear powers of the field x which is the case for an interacting model with $g \neq 0$.

The variation of the fields under this second (nilpotent) BRST symmetry is

$$\begin{aligned}
\delta_{(2)} x_i &= \xi \overline{\psi}_i \\
\delta_{(2)} \psi_i &= \xi \left(B_i - 2P'_i(x)\right) \\
\delta_{(2)} \overline{\psi}_i &= 0 \\
\delta_{(2)} B_i &= 2\xi P''_i \overline{\psi}\,.
\end{aligned}$$

In the continuum where $S_{(1)} = S_{(2)}$ the action of supersymmetric quantum mechanics would then possess *two* BRST invariances. On the lattice if we choose $S_{(1)}$ as action we no longer have $\delta_{(2)}$ as a symmetry (except for a free theory) and vice versa. Thus at the *classical* level discretization on a lattice necessarily breaks one of the continuum symmetries. However, we will see that this symmetry is *restored* at the quantum level with lattice Ward identities corresponding to both $\delta_{(1)}$ and $\delta_{(2)}$ being satisfied exactly for arbitrary lattice spacing. Indeed we will prove that there exists a one parameter family of lattice actions in which all of the BRST Ward identities are satisfied with *no* cut-off effects. This feature is crucially dependent on the existence of this topological symmetry.

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4. Continuum topological field theories

In this section we give a brief review of some of the general features of continuum topological field theories [5, 7]. Such theories are formulated on a n-dimensional manifold equipped with a metric $g_{\mu\nu}$. On this space there exists a set of fields Φ and an action $S(\Phi)$. Typically these theories contain operators or topological observables $O_\beta(\Phi)$ with the property that their expectation values are metric independent

$$\frac{\delta}{\delta g^{\mu\nu}}\langle O_{\beta_1}\cdots O_{\beta_2}\rangle = 0$$

One way to guarantee this corresponds to the case in which there exists a nilpotent symmetry δ such that

$$\delta O_\beta = 0\,, \qquad T_{\mu\nu} = \delta G_{\mu\nu}\,.$$

The energy-momentum tensor $T_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}}S(\Phi)$. These conditions lead to the following expressions for the expectation values of topological variables.

$$\frac{\delta}{\delta g^{\mu\nu}}\langle O_{\beta_1}\ldots O_{\beta_2}\rangle = -\int D\Phi\delta\left(O_{\beta_1}\ldots O_{\beta_2}G_{\mu\nu}e^{-S(\Phi)}\right) = 0\,.$$

Here we have also assumed that the measure is invariant under the nilpotent symmetry and that the observables themselves do not contain the metric explicitly. Theories constructed in this way are called Witten type or cohomological topological quantum field theories. Typically the nilpotent symmetry δ is realized as a BRST symmetry arising from gauge fixing some underlying local shift symmetry. In such theories

$$S(\Phi) = \delta\Lambda(\Phi)\,.$$

Actually, as we stated earlier, the requirement that physical states are invariant under the gauge symmetry imposes additional constraints on the physical Hilbert space — namely that states of the topological field theory are annihilated by the BRST charge.

This result eq. (4) is very important as it guarantees that topological observables and the partition function itself can be computed *exactly* in the semi-classical limit. To see this introduce a parameter ϵ playing the role of Planck's constant in the definition of a topological expectation value and examine the variation of that expectation value under variation of ϵ

$$\frac{\partial}{\partial\epsilon}\langle O_{\beta_1}\ldots O_{\beta_2}\rangle = \frac{1}{\epsilon^2}\int D\Phi\delta\left(O_{\beta_1}\ldots O_{\beta_2}\Lambda(\Phi)\right)e^{-\frac{1}{\epsilon}S} = 0\,.$$

Thus topological observables may be computed exactly in the semi-classical approximation $\epsilon\to 0$. This semi-classical exactness may be translated into an independence of topological observables on coupling. To see this consider an action of the form

$$S = S_0(\Phi) + g\Phi^n$$

where the quadratic terms are contained in $S_0(\Phi)$ and we allow for a generic interaction term. If we rescale the fields using $\Phi\to\sqrt{\epsilon}\Phi$ it is easy to see that topological observables computed in an ensemble with Planck constant ϵ and coupling g is equivalent to the same observable computed in the ensemble $\epsilon = 1$ and $g' = g\epsilon^{n/2-1}$ The limit $\epsilon \to 0$ now corresponds to $g' \to 0$ in the latter ensemble and hence the expectation value can be computed in the free field limit. For the case $O = 1$ this implies that the partition function itself is independent of g. This, of course, is also the property enjoyed by models with a local Nicolai map and makes plausible the conjecture that models possessing a local Nicolai map contain within them a topological symmetry.

A trivial set of topological observables correspond to operators of the form

$$O = \delta O' .$$

While trivial in a true topological sense (their expectation value vanishes trivially on account of the nilpotent nature of δ) they will be of crucial importance in the parent supersymmetric theory since they yield the supersymmetric Ward identities.

5. Relation between lattice and continuum theories

In section 3 we discussed a lattice theory of supersymmetric quantum mechanics and showed that it was possible to choose an action which reproduced the continuum partition function exactly (up to a parameter independent multiplicative constant). The simplest way to see this utilizes the gauge $\alpha = 0$ in which the partition function reduces to a sum over the critical points of the potential $P'(x) = 0$. The solution of this equation is identical in both lattice and continuum theories. Actually, the gauge $\alpha = 0$ can be used for an arbitrary topological observable and implies that the expectation values for all such observables only depend on the properties of the classical solution to the field equations. This prompts us to guess that it should be possible to forge an explicit connection between the lattice and continuum theories useful for the computation of such observables. We will now show that indeed this is the case.

One standard way to relate a lattice theory to an underlying continuum theory derives from the renormalization group. The lattice field at some point is constructed by averaging the continuum field over a neighborhood of that point. This averaging or *blocking* procedure may be accomplished by convolving the continuum field with a blocking function. Typically, the precise shape of the blocking function is not important for long distance physics. The lattice or *block* field which results from this operation is usually a non-local function of the continuum fields. However, this need not be the case for a topological field theory. If we are only concerned with the computation of topological observables we are at liberty to block the continuum fields in an *arbitrary* background metric. If this metric is then chosen carefully we can arrange for the block fields to be related to the continuum fields in a *completely local* manner.

Let us examine this in the case of supersymmetric quantum mechanics. The bosonic part of the continuum action for an arbitrary one-dimensional metric written in terms of the einbein $e(t)$ takes the form (we have integrated out the auxiliary field $B(t)$)

$$S = \int dt\, e(t) \left[\frac{1}{e(t)} \frac{dx}{dt} + P'(x) \right]^2 .$$

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Define now a scalar block field $x^B(t)$ *in the continuum* as a convolution over the original field $x(t)$ using a blocking function $B^-_\beta(t)$

$$x^B(t) = \int dt'\, e(t') B^-_\beta\left(t - t'\right) x(t') \,. \tag{5.1}$$

We will choose the blocking function $B^-_\beta(t)$ to be

$$B^-_\beta(t) = \frac{1}{2a}\left[L_\beta\left(t + \delta\right) - L_\beta\left(t - a + \delta\right)\right].$$

We require that the function $L_\beta(t)$ tend to the step function for large β

$$\lim_{\beta\to\infty} L_\beta(t) = \theta(t) \,.$$

This choice of $B^-_\beta(t)$ ensures that for large β the blocked field at point t contains contributions from all points within a cell defined by $-\delta \le t \le (a - \delta)$. Furthermore, we will require the parameter $\delta \to 0^+$ at the end of the calculation. A concrete example of such a function is given by $L_\beta(t) = \tanh \beta t$. To capture the structure of the lattice theory we will choose an associated "lattice" metric given by $e(t) = e_\beta(t)$ where

$$e_\beta(t) = \sum_{n=1}^{N} \frac{a}{A(\beta)} L'_\beta(t - na)$$

where the sum runs over a finite set of N points with "lattice spacing" a. The constant $A(\beta)$ is chosen so that

$$\int dt e_\beta(t) = Na$$

and we assume the continuum theory is defined over a circle with circumference Na. Notice that

$$\lim_{\beta\to\infty} L'_\beta = \lim_{\beta\to\infty} \frac{dL_\beta}{dt} = A_L \delta(t)$$

where $A_L = A(\infty)$ is just some numerical coefficient. Returning to eq. (5.1) we can now compute the block field explicitly

$$x^B(t) \simeq \sum_{n=1}^{N} \frac{a}{A(\beta)} x(na) B^-_\beta(t - na) \quad \text{for large } \beta \,.$$

In the limit $\beta \to \infty$ this relation yields the result

$$\lim_{\beta\to\infty} x^B(t) = \sum_{n=1}^{N} x(na) \frac{1}{2A_L}\left[\theta(t - na + \delta) - \theta(t - (n+1)a + \delta)\right].$$

That is, the continuum block field $x^B(t)$ is constant within each unit cell of the lattice changing its value only on passing from one cell to the next. To proceed further it is necessary to compute its derivative.

$$\begin{aligned}\frac{dx^B(t)}{dt} &\simeq \sum_{n=1}^{N} \frac{a}{A(\beta)} x(na) \frac{dB_\beta^-}{dt}(t-na) \\ &\simeq \frac{1}{2A(\beta)} \sum_{n=1}^{N} x(na) \left[L'_\beta(t-na+\delta) - L'_\beta(t-(n+1)a+\delta)\right].\end{aligned}$$

Notice that this derivative does indeed vanish in the limit $\beta \to \infty$ for any point within a cell. To compute the action evaluated on a block configuration in this background we need to compute $\frac{1}{e_\beta}\frac{dx^B}{dt}$. For any point within the cell $-\delta + na \leq t \leq (n+1)a - \delta$ the leading contribution at large β is seen to be

$$\begin{aligned}\frac{1}{e_\beta}\frac{dx^B}{dt} \simeq{}& \frac{1}{2aA(\beta)} \times \\ &\times \left(f_\beta(z)\left[2x(na) - x((n+1)a) - x((n-1)a)\right] + \left[x((n+1)a) - x((n-1)a)\right]\right)\end{aligned}$$

where

$$f_\beta(z) = \left(\frac{L'_\beta(a-z) - L'_\beta(z)}{L'_\beta(z) + L'_\beta(a-z)}\right)$$

with $z = t - na$ and we have set δ to zero for simplicity. This in turn reduces to

$$\begin{aligned}\lim_{\beta\to\infty} \frac{1}{e_\beta(t)}\frac{dx^B}{dt} ={}& \sum_{n=1}^{N} \frac{1}{2aA_L}\left[x(na) - x((n-1)a)\right] \times \\ &\times \left[\theta\left(t - \left(n - \frac{1}{2}\right)a\right) - \theta\left(t - \left(n + \frac{1}{2}\right)a\right)\right].\end{aligned}$$

Thus the derivative of a block function in such a background geometry at large β is constant now within a unit cell of the *dual lattice*. These properties allow us to compute the integral of an arbitrary function of the block field $x^B(t)$ and its first derivatives. An example is the bosonic part of the continuum action S_B. This becomes

$$\lim_{\beta\to\infty} S_B = \sum_{n=1}^{N} a\left[\frac{1}{2A_L a} D^-_{nm} x^B_m + P'_n\left(x^B\right)\right]^2 \tag{5.2}$$

where the bosonic action now *only depends on the blocked fields at the lattice points* and we use the obvious notation $x^B(na) \equiv x^B_n$. The most striking thing about this block action S_B is that it coincides with the bosonic part of the lattice action $S_{(1)}$ discussed earlier if we identify the lattice field as the blocked continuum field evaluated on a lattice point[1]. Notice also that the lattice theory we arrive at in this manner automatically incorporates an $r = 1$ Wilson mass term to remove potential lattice doublers (since $D^S - m_W = D^-$).

[1] a finite rescaling of the field is also needed to make the correspondence exact - but this in turn is equivalent to a rescaling of the coupling which will not affect topological observables

Notice that the backward difference operator D^- arises directly from our choice of blocking function B_β^-. If we had made the equally valid choice

$$B_\beta^+ = \frac{1}{2a}\left[L_\beta\left(x+a-\delta\right) - L_\beta\left(x-\delta\right)\right]$$

we would have arrived at a block action of the same form as in eq. (5.2) but with D^- replaced by the forward derivative D^+.

Generalization to include the fermionic sector is straightforward and leads to the conclusion that the total action when evaluated on block configurations as defined by eq. (5.1) goes over into the full lattice action $S_{(1)}$ described in section 3.

This similarity between the blocked continuum theory at large β and the lattice model described in section 3 is intriguing but by itself does not yet guarantee the exact equivalence of continuum and lattice theories. To complete the correspondence between continuum and lattice theories, and to show that for topological observables the continuum expectation values are just equal to their lattice counterparts, we still need to argue that *only* block fields need to be taken into account inside continuum path integrals as $\beta \to \infty$. We offer two arguments for this.

First consider the continuum boson kinetic term in terms of the original fields

$$S_{\mathrm{K}} = \int dt \frac{1}{e_\beta}\left(\frac{dx}{dt}\right)^2 .$$

As $\beta \to \infty$ it is clear that away from the lattice points $t = na$ the action starts to diverge due to the presence of $e_\beta(t)$ in the denominator. Furthermore, it is clear that this effect yields an exponential suppression of any field configuration $x(t)$ in which the field changes rapidly within a cell. Indeed, we can argue that that the only configurations that survive in the path integral are those in which $\frac{dx}{dt} \sim e^{-\beta\delta t/2}$ for δt away from a lattice point. Thus, in the limit $\beta \to \infty$ only the block boson fields survive in the path integral. The relations $\delta_{(1)}x = \psi$ and $\delta_{(2)}x = \overline{\psi}$ then ensure that, in the absence of topological symmetry breaking, only block fermion fields need to be considered for large β.

Our second argument is formal but more general. We are at liberty to consider the blocking transformation given in eq. (5.1) for *any* finite β as corresponding to a simple change of variables in the continuum partition function. Let us write this transformation for a generic field $\Phi = \{x, B, \psi, \overline{\psi}\}$ using a compact notation

$$\Phi^B = K_\beta \Phi$$

where the kernel K_β is shorthand for

$$K_\beta\left(t, t'\right) = e_\beta(t) B_\beta^-\left(t - t'\right)$$

and we leave implicit the integrals defining the convolution of K_β with any field Φ. In terms of these new variables the partition function becomes

$$Z = \int D\Phi^B J(\beta) e^{-S\left(K_\beta^{-1}\Phi^B\right)}$$

where the jacobian $J(\beta)$ is *independent* of the fields since the transformation is linear. Indeed, the linearity of this mapping implies that each of the BRST transformations for the original continuum fields yields identical transformations of the block fields. For example,

$$\begin{aligned}
\delta_{(1)} x^B &= \psi^B \xi \\
\delta_{(1)} \overline{\psi}^B &= B^B \xi \\
\delta_{(1)} \psi^B &= 0 \\
\delta_{(1)} B^B &= 0\,.
\end{aligned}$$

Thus the BRST operators remain nilpotent on the block fields. We assume that the inverse operator K_β^{-1} exists. For generic values of β the resulting action $S\left(K_\beta^{-1}\Phi^B\right)$ will be a complicated and non-local function of the block fields. However, it has one crucial property — it will still be invariant under a BRST variation of the block fields. Indeed, we can construct topological observables from the block fields just as before

$$O^B_{\text{top. obs.}} = \delta_{(i)} T^B\left(\Phi^B\right) \tag{5.3}$$

for any function $T(\Phi^B)$ and either of the symmetries $i = 1, 2$. Now imagine taking the limit $\beta \to \infty$. By examining the form of the kernel it is easy to see that in this limit the block field becomes an eigenvector of the kernel

$$\lim_{\beta\to\infty} K\Phi^B = \frac{1}{2A_L}\Phi^B\,.$$

Actually we must be careful here — as $\beta \to \infty$ it is clear that the operator K develops a set of zero modes f_l where $Kf_l = 0$ corresponding to functions which are zero at the lattice points but are otherwise unrestricted. These can be safely ignored only by considering the form of the boson kinetic term which ensures that they are wholly suppressed in the limit $\beta \to \infty$ as we argued above. Ignoring these modes (which are not present at finite β) leads us to conclude that $K_\beta^{-1}\Phi^B \to 2A_L\Phi_B$ and thus the action need only be evaluated for the block fields

$$\lim_{\beta\to\infty} S\left(K_\beta^{-1}\Phi^B\right) = S\left(2A_L\Phi^B\right).$$

Thus in this limit the partition function factorizes into a piece determined by the block fields at a finite set of "lattice" points labeled by an integer n together with an integral over all points lying in cells of the "lattice"

$$\lim_{\beta\to\infty} Z = J(\infty)\int D\Phi^B_{\text{cells}}\int D\Phi^B_n e^{-S(2A_L\Phi^B_n)}\,.$$

Throwing away this irrelevant (infinite) multiplicative constant we see that we have arrived at our original lattice action determined by a finite set of variables! It is not hard to show that in the large β limit the observables given in eq. (5.3) go over into the lattice Ward identities. Since our lattice theory has been obtained by a process of continuous deformation of the continuum theory we expect the resultant lattice theory to preserve the values of *all* continuum topological observables. Notice though that at $\beta = \infty$ the continuum block fields are discontinuous functions. Thus we should not be surprised if the Leibniz rule fails when applied to functions of such fields and seeming total derivative terms do not vanish in the block action.

In the language of the renormalization group we have found that the lattice action of section 3 is a *perfect* lattice action for the computation of topological observables — it yields cut-off independent predictions for the corresponding expectation values. Thus the lattice theory should contain a set of exact Ward identities corresponding to any combination of supersymmetries which yields a continuum topological symmetry. The arguments presented above, while not rigorous, are well supported by the numerical results, as we will show in the next section.

6. Ward identities in supersymmetric quantum mechanics

We have tested these ideas by simulating the model given by the action $S_{(1)}$ using a potential of the form given in eq. (3.1) which contains a mass term m plus single coupling g. Details of our accelerated HMC algorithm are given in [8]. The results were obtained for lattice parameters $m = 0.25$, $g = 0.0625$ on an $L = 4$ site lattice and utilized 10^8 HMC trajectories. Since the dimensionless interaction strength is given by the parameter $g/m^2 = 1$ this choice of parameters corresponds to a strongly coupled theory on a coarse lattice. As such it should easily reveal any symmetry breaking in the lattice theory. We examined the first non-trivial Ward identities corresponding to the expectation values $\langle \delta_{(1)} O_1 \rangle$ and $\langle \delta_{(2)} O_2 \rangle$ for $O_1 = x_i \overline{\psi}_j$ and $O_2 = x_i \psi_j$. These take the form

$$
\begin{aligned}
\left\langle x_i N_j^{(1)} \right\rangle + \left\langle \psi_i \overline{\psi}_j \right\rangle &= 0 \\
\left\langle x_i N_j^{(2)} \right\rangle + \left\langle \overline{\psi}_i \psi_j \right\rangle &= 0
\end{aligned}
$$

The results are shown in tables 1 and 2. For an exact Ward identity the sum of the bosonic and fermionic contributions across any row should cancel. Since the action $S^{(1)}$ is invariant under the $\delta_{(1)}$-symmetry it should be no surprise that the corresponding Ward identity is satisfied. What is, on the surface, much more surprising is that the Ward identity corresponding to the $\delta_{(2)}$-symmetry is also satisfied to within the (small) statistical errors. This despite the fact that the lattice action is not invariant under this symmetry. Of course this is just the result one would expect on the basis of the arguments of the last section since $\delta_{(2)}$ generates an independent topological symmetry of the continuum theory.

t	$\langle x(0) N^{(1)}(t) \rangle$	$\langle \psi(0) \overline{\psi}(t) \rangle$
0	0.8895(11)	−0.8898(3)
1	0.6152(10)	−0.6155(3)
2	0.4294(11)	−0.4295(3)
3	0.3024(11)	−0.3028(3)

Table 1: Ward identity for $\delta_{(1)}$-symmetry at $g = 0.0625$, $m = 0.25$ and $L = 4$.

t	$\langle x(0) N^{(2)}(t) \rangle$	$\langle \overline{\psi}(0) \psi(t) \rangle$
0	−0.8895(11)	0.8898(3)
1	−0.3016(11)	0.3028(3)
2	−0.4294(11)	0.4295(3)
3	−0.6160(10)	0.6155(3)

Table 2: Ward identity for $\delta_{(2)}$-symmetry at $g = 0.0625$, $m = 0.25$ and $L = 4$.

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Since the lattice actions $S_{(1)}$ and $S_{(2)}$ differ only by the cross term $C(g)$ and both correspond to theories which retain all the continuum topological symmetry for any coupling g we are led to conclude that perturbations about either lattice theory via such a term will not affect topological observables. The simplest such observable is the partition function itself. To check this we simulated the same model as before *except* we added the cross term $C(g)$ to the action with arbitrary coupling γ.

γ	$\langle S\rangle$
0	3.9997(41)
1	4.0003(39)
2	4.0003(40)

Table 3: $\langle S\rangle$ vs γ for $g = 6.25$, $m = 2.5$ and $L = 4$.

$$S = S_{(1)} - \gamma C(g)$$

Table 3 shows the mean value of the action $\langle S\rangle$ on an $L = 4$ site lattice using now $g = 6.25$, $m = 2.5$ for three values of the coupling γ. Notice the data we show here corresponds to even coarser lattices than before with the same value for the interaction parameter g/m^2. Here, the measured action includes contributions from the scalars and pseudofermions and, as detailed in [2], can be shown to take the value

$$\langle S\rangle = L + g\frac{\partial}{\partial g} Z\left(g\right).$$

Thus, a partition function constructed from a BRST invariant theory which is independent of the coupling g should yield $\langle S\rangle = L$. At $\gamma = 0$ and $\gamma = 2$ (corresponding to lattice action $S^{(2)}$) this is indeed the case. What is more surprising is that it appears to be also true at $\gamma = 1$ where the *classical* lattice action is not invariant under either the $\delta_{(1)}$ or $\delta_{(2)}$ symmetries. We have checked this result for other values of γ and for different masses and couplings with identical results. To reinforce this point we show in tables 4 and 5 the same two Ward identities as before computed in the $\gamma = 1$ ensemble.

Again both Ward identities are satisfied within statistical errors. Thus our numerical results lend strong support to the results of section 5

7. Complex Wess-Zumino model in two dimensions

These arguments can be extended to the complex Wess Zumino model with $N = 2$ supersymmetry in two dimensions. A lattice formulation of this model based on a discretization of the Nicolai map was studied in [9] and [10]. More recently the Nicolai map was extended to Ginsparg-Wilson fermions in [11]. In [12] a hamiltonian approach was used to investigate the same model.

t	$\langle x(0)N^{(1)}(t)\rangle$	$\langle\psi(0)\overline{\psi}(t)\rangle$
0	0.23102(37)	−0.23110(6)
1	0.05367(15)	−0.05334(3)
2	0.01251(14)	−0.01232(1)
3	0.00263(11)	−0.002848(3)

Table 4: Ward identity for $\delta_{(1)}$-symmetry at $g = 6.25$, $m = 2.5$ $L = 4$ and $\gamma = 1.0$.

t	$\langle x(0)N^{(2)}(t)\rangle$	$\langle\overline{\psi}(0)\psi(t)\rangle$
0	−0.23102(38)	0.23110(6)
1	−0.00265(11)	0.002848(3)
2	−0.01251(14)	0.01232(1)
3	−0.05365(15)	0.05334(3)

Table 5: Ward identity for $\delta_{(2)}$-symmetry at $g = 6.25$, $m = 2.5$ $L = 4$ and $\gamma = 1.0$

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In [2] we showed that this model can be derived from the simple model eq. (2.1) if one allows the original index to represent both a lattice coordinate in two dimensional euclidean space and a two-component internal degree of freedom. The resulting theory contains Dirac fermions coupled to a complex scalar field ϕ and in the continuum possesses $N = 2$ supersymmetry. This formulation of the model has the merit of exhibiting clearly its relationship to a continuum topological field theory (the connection between the parent supersymmetric field theory and its topological projection is discussed in [5] and references therein)

In [2] we showed that there were four possible choices for the scalar lattice action

$$S(\phi) = \frac{1}{2}\eta^{(a)}\overline{\eta^{(a)}}$$

each of which exhibited an exact fermionic symmetry and corresponded to four inequivalent local Nicolai maps $\eta^{(a)}(\phi)$ with $a = 1\ldots 4$. Each of these fermionic symmetries is now seen to be a BRST symmetry corresponding to four different quantizations of an underlying complex scalar field theory possessing a local shift symmetry. The distinct Nicolai maps simply correspond to different gauge choices for the scalars possessing the *same* Fadeev-Popov determinant. The explicit forms of these maps are (see [2])

$$\begin{aligned}
\eta^{(1)} &= D^S_z\overline{\phi} + W'(\phi) \\
\eta^{(2)} &= D^S_z\overline{\phi} - W'(\phi) \\
\eta^{(3)} &= D^S_z\phi - W'(\overline{\phi}) \\
\eta^{(4)} &= D^S_z\phi + W'(\overline{\phi})\,.
\end{aligned}$$

They come in two groups of two corresponding to complex conjugation of the scalar field ϕ. Notice that here we are again using the notation of section 3 in which kinetic terms are written in terms of symmetric difference operators and wilson masses are added into the potential terms. Since $r = 1$ this is entirely equivalent to the block language of forward and backward derivatives and local potential terms. The actions corresponding to the first two of these maps differ only by a cross term of the form

$$C_1 = 2Re\left(D^S_z\overline{\phi}W'(\overline{\phi})\right).$$

Similarly the actions corresponding to the third and fourth Nicolai maps would differ only by another cross term of the same form but with $\phi \to \overline{\phi}$

$$C_2 = 2Re\left(D^S_z\phi W'(\phi)\right).$$

As before we can attempt to derive these lattice models by blocking the continuum field theory. To do this we need generalizations to two dimensions of the lattice metric and blocking functions. The following choices appear to suffice (for simplicity we parametrize the bosons in terms of two real fields here)

$$g(\sigma_1,\sigma_2) = diag\left(e^2_\beta(\sigma_1), e^2_\beta(\sigma_2)\right).$$

The components of this diagonal matrix are just the squares of the functions $e_\beta(t)$ introduced in section 5, each now being a function of the corresponding coordinate σ_i. Similarly the two-dimensional blocking function $B^{\rm 2d}_\beta(x)$ can be taken to be just the product of one-dimensional blocking functions, for example,

$$B^{\rm 2d}_\beta(\sigma_1,\sigma_2) = B^-_\beta(\sigma_1)B^-_\beta(\sigma_2)\,.$$

By following the same procedure as for supersymmetric quantum mechanics we can derive a lattice action by blocking out of the continuum a topological theory built from any of the Nicolai maps detailed above. Furthermore, topological observables such as the Ward identities associated with any of these continuum BRST symmetries should not depend on the couplings to any cross terms of the form of C_1 or C_2 since such operators interpolate between equivalent topological field theories. Thus we predict that the following simple lattice action will yield exact Ward identities for all four symmetries at the quantum level even though it possesses *none* of these symmetries at the classical level

$$S = \sum_z \frac{1}{2}\left(D^S_z\overline{\phi}D^S_{\overline{z}}\phi + W'(\phi)W'(\overline{\phi})\right) + \ln\det M \tag{7.1}$$

where M is the fermion operator corresponding to, for example, the choice $\eta^{(1)}$ (the determinant of M is independent of which map is used).

We have checked these conclusions by measuring the four simplest non-trivial Ward identities corresponding to these BRST symmetries for the action given by eq. (7.1) on an 8×8 site lattice with $g/m = 1$ and lattice mass $m = 0.625$. The detailed form of these was derived in [2]. For completeness we list them again here

$$\begin{aligned}
0 &= \left\langle \psi^\alpha_i\overline{\psi}^\beta_j\right\rangle + \left\langle N^\beta_j x^\alpha_i\right\rangle\\
0 &= \left\langle i\gamma_3^{\alpha\gamma}\psi^\gamma_i\overline{\psi}^\beta_j\right\rangle + \left\langle i\gamma_3^{\beta\gamma}\overline{N}^\gamma_j x^\alpha_i\right\rangle\\
0 &= \left\langle \gamma_1^{\alpha\gamma}\overline{\psi}^\gamma_i\psi^\beta_j\right\rangle + \left\langle Q^\beta_j x^\alpha_i\right\rangle\\
0 &= \left\langle \gamma_2^{\alpha\gamma}\overline{\psi}^\gamma_i\psi^\beta_j\right\rangle + \left\langle i\gamma_3^{\beta\gamma}\overline{Q}^\gamma_j x^\alpha_i\right\rangle .
\end{aligned}$$

Notice that these expressions involve not the original complex boson fields $\phi, \eta^{(i)}$ but their real counterparts eg. $\phi = x_1 + ix_2$ and $\eta^{(1)} = N_1 + iN_2$ with similar expressions for $\eta^{(2)}$, $\eta^{(3)}$ and $\eta^{(4)}$ in terms of $\overline{N}_\alpha$, Q_α and $\overline{Q}_\alpha$ respectively where $\alpha = 1, 2$. Figure 1 shows a plot of the (11)-component of the first Ward identity corresponding to the $\delta_{(1)}$-symmetry. For simplicity all the correlators we exhibit correspond to timeslice averaged fields. Both bosonic and fermionic contributions are shown together with their sum W^1 corresponding to the Ward identity. Since W^1 is zero within statistical errors it is clear that the Ward identity is satisfied even though the starting lattice action given by eq. (7.1) does not possess this symmetry exactly. We find this situation to be true for all the Ward identities — as a further example, figure 2 shows the (12)-component of the 2nd Ward identity. Here W^2, the sum of a fermionic and bosonic correlators, is again seen to be close to zero over the entire range of time. We have checked these conclusions for a variety of lattice parameters and sizes with the same result.

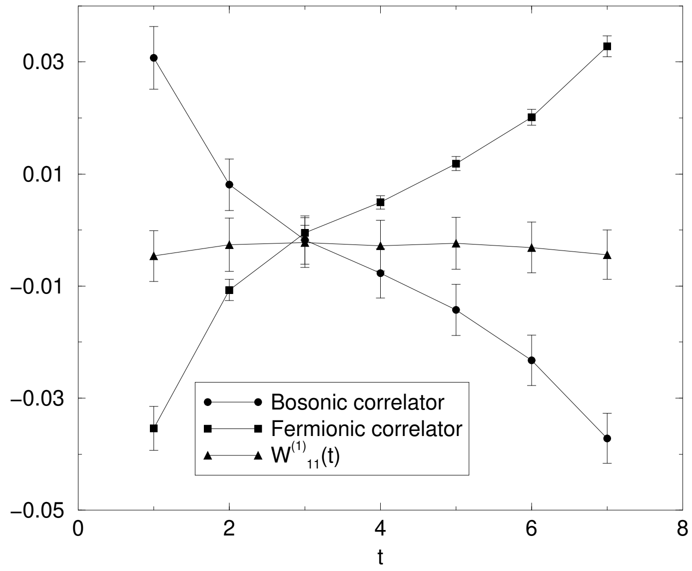

Figure 1: Ward identity corresponding to $\delta_{(1)}$-symmetry with $m = 0.625$, $g = 0.625$ and $L = 8$.

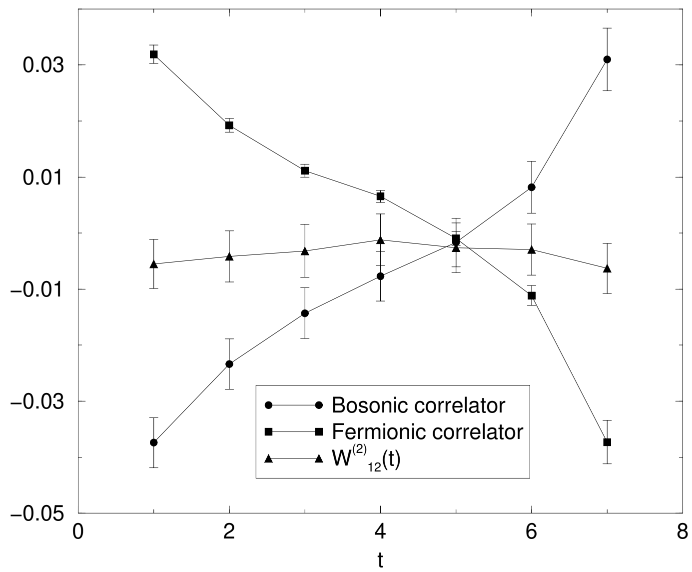

Figure 2: Ward identity corresponding to $\delta_{(2)}$-symmetry with $m = 0.625$, $g = 0.625$ and $L = 8$.

In [13] it was argued that the presence of a local Nicolai map in the lattice model played no essential role in determining the renormalization properties of the theory. However, the existence of a local Nicolai map can now be viewed as a consequence of the topological character of the continuum theory. Thus we would claim that the benign U.V behavior of the lattice model in eq. (7.1) is intimately connected to the existence of an exact Nicolai map in the associated continuum model.

8. Discussion

It is well known that certain one and two-dimensional non-gauge models can be discretized in such a way as to preserve a single parameter fermionic symmetry. In two cases - supersymmetric quantum mechanics and the complex Wess-Zumino model in two dimensions we show that this lattice invariance (and the associated existence of a local Nicolai map) follow from an underlying topological structure in the continuum field theories. Indeed certain Ward identities of the supersymmetric model just correspond to trivial topological observables in the topological field theory.

As we have stressed before, the topological model is strictly a subsector of the parent supersymmetric field theory corresponding to a projection onto states annihilated by the BRST charge. The latter include any supersymmetric vacuum states plus states created by acting on some state with the BRST charge. The latter include the operators corresponding to trivial topological observables. In this paper we argue that, at least for these low-dimensional models where all the original supersymmetries can be thought of as topological, we may also imagine relaxing this constraint completely and consider the models as bona fide discretizations of the parent supersymmetric field theories. The numerical results showing the degeneracy of boson and fermion masses in both supersymmetric quantum mechanics [4] and the Wess-Zumino model [2] lend strong support to this idea.

Furthermore, we argue that these lattice models can be arrived at by blocking out of the continuum in a carefully chosen background geometry. Such a procedure will generate a *perfect* lattice action for the computation of topological observables and hence guarantees an absence of cut-off effects in the lattice Ward identities. For the models considered here which contain two and four topological symmetries respectively this connection of the lattice model to the continuum guarantees that *all* the supersymmetric Ward identities are satisfied exactly on the lattice. This may even be true for lattice actions which are *not* classically invariant under the some of the supersymmetries.

The crucial ingredient in topological field theories of this type is that they contain a scalar nilpotent charge arising from BRST quantizing an underlying bosonic theory. In terms of the original supersymmetry this charge is obtained by taking particular linear combinations of the supercharges. This latter procedure requires that the original theory possess extended supersymmetry. The question that remains is how much of this structure survives in higher dimensions and in the presence of gauge symmetry. The procedure to obtain a nilpotent charge from a set of supercharges is termed *twisting* and is a well known method to obtain topological field theories from theories with extended supersymmetry in higher dimensions [14]. Indeed, Donaldson-Witten theory was the first topological field theory to be constructed and corresponds to twisting $N = 2$ super Yang Mills theory. In general, in higher dimensions, only a fraction of the original supersymmetries can be reinterpreted as yielding BRST charges and it is only this fraction that we can hope to preserve on the lattice. Nevertheless, it is tempting to try and use this topological field theory interpretation to construct lattice models containing a residual element of supersymmetry. The procedure of blocking out of the continuum in the presence of a carefully chosen "lattice" metric may prove very fruitful in this regard. With regard to this the presence of more than one topological symmetry may be necessary in order to establish an explicit connection between the lattice and continuum theories.

While we think that this approach deserves further study it will clearly be problematic — the lattice theory, to duplicate the structure of the continuum theories, must contain non-compact fields, and so such theories will necessarily break gauge invariance at the classical level. To utilize this formalism it will then be necessary to show that the existence of this residual supersymmetry can constrain the radiative corrections sufficiently to eliminate dangerous gauge-violating counterterms. Also, in four dimensions, it is not clear that we can consistently drop the projection to the vacuum states associated with the true topological theory without inducing negative norm states in the spectrum of the lattice theory.

Additionally, local Nicolai maps are known to exist for a number of other models [15] not all of which have $N = 2$ supersymmetry. This is intriguing as it points to a possible hidden topological structure in those models which, if elucidated, may itself help with the problem of studying such models on lattices.

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